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Analytical Solutions for Lag/Lead and General Second Order Compensator Design Problems

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Abstract- **This paper presents a simple method for determining analytical solutions to lag/lead cascade and general second order compensator design problems in the frequency domain. Only linear and quadratic equations are used in the proposed approach. Results obtained here can be used to eliminate conventional graphic-based trial-and-error method used in past and contemporary control textbooks, which are tedious and time-consuming, and rewrite them into simple and analytical non-trial-and-error steps.**

Index Terms—**Lag/lead compensators, cascade compensation, trial-and-error, analytic solution.**

1. INTRODUCTION

Although control system design has made tremendous progress over the last three decades, and advanced mathematic concepts, methods, and tools are used extensively in modern control theory, many industrial control systems are still designed by simple procedures in the frequency domain, especially the PID and lag/lead cascade compensation. Those classical approaches are very valuable due to their simplicity, particularly when no analytical plant models are unknown and only experimental data in frequency domain are available. Since the cascade compensation was first introduced, the determination of compensators has always been carried out by the conventional graphic-based trial-and-error design method, as one can see from all available past and contemporary textbooks in control [11-20]. Usually, an appropriate compensator can be obtained only after many trials and errors, usually a tedious and time-consuming process.

From the control educational point of view, many consider the frequency domain design as "physics" of communication, control, and signal processing systems, especially when mathematics plays a more and more important role in teaching, research, and application of those fields in the modern age. This is quite natural and justified since frequency domain design methods expose students and engineers to variables and concepts that can be directly related to phenomena and quantities in the physical world, not just some heavy doses of matrices, equations, and their manipulations in mathematics. This is part of the reason why frequency domain design is still play a fundamental role in both engineering education and industrial applications.

However, due to the nature of conventional trial-and-error graphical techniques currently used by almost all available textbooks [11-20], learning and use of cascade compensation are still a time consuming process, and become more serious a problem in teaching and applications now since computers are used for everything and students are pressed for time in learning new and old subjects. Starting from Wakeland and Mitchell in 1970s [1-4], Yeung, et al and Wang in 1990s [5-10], various efforts have been made to develop analytical or computer-aided design procedures for lag and lead compensation with limited and partial success.

In this paper, we present a simple method for finding analytical solutions to lag/lead cascade and general second order compensation design problems. In this approach, using Euler formula for complex numbers, only linear equations are involved for solving unknown design parameters, no nonlinear or transcendental equations as in previous works. The proposed analytical solutions can be used as the basis to eliminate and rewrite traditional trial-and-error procedures in control textbooks into analytical non-trial-and-error steps for determining cascade compensators in frequency domain.

2. DESIGN SPECIFICATION FOR COMPENSATORS

Fig. 1 shows the diagram for a feedforward compensation system, where Gp(s) and Gc(s) represent the given plant and the compensator to be designed, respectively. To facilitate the derivation, let us rewrite compensator as,

$$G_c(s) = K_c \overline{G}_c(s) \quad (1)$$

where $\overline{G}_c(0) = 1$. As usual, K_c is assumed to have been determined from the steady-state accuracy specification using the final value theorem for Laplace transforms [6, 10-16]. Thus, $\overline{G}_c(s)$ is the part of the compensator that needs to be determined.

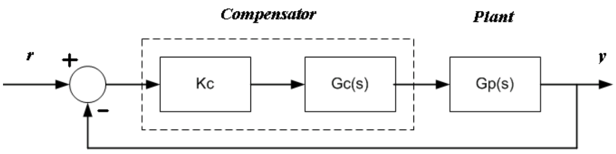

Fig. 1: A Typical Feedforward Compensation System

For a desired gain margin GM at a phase crossover frequency ω_1, it follows that,

$$\left|G_c(j\omega_1)G_p(j\omega_1)\right|_{dB} = -GM, \quad \angle\{G_c(j\omega_1)G_p(j\omega_1)\} = -180^\circ$$

where $\left|(\cdot)\right|_{dB} = 20\log_{10}(\cdot)$, or in terms of $\overline{G}_c$ only,

$$\left|\overline{G}_c(j\omega_1)\right| = c_1, \quad \angle\overline{G}_c(j\omega_1) = p_1 \quad (2)$$

where

$$c_1 = \exp(-GM/20)/\left|K_c G_p(j\omega_1)\right|, \quad p_1 = -\angle G_p(j\omega_1) - 180^\circ \quad (3)$$

Similarly, for a desired phase margin PM at a gain crossover frequency ω_2, we have,

$$\left|G_c(j\omega_2)G_p(j\omega_2)\right|_{dB}=0,\ \angle\{G_c(j\omega_2)G_p(j\omega_2)\}=PM-180^{\circ}$$

or in terms of $\overline{G}_c$ only,

$$\left|\overline{G}_c(j\omega_2)\right|=c_2,\ \angle\overline{G}_c(j\omega_2)=p_2 \quad (4)$$

where

$$c_2=1/\left|K_cG_p(j\omega_2)\right|,\ p_2=PM-\angle G_p(j\omega_2)-180^{\circ}$$

Using Euler's Formula, we can rewrite Eqs. (2) and (4) into the following form,

$$\overline{G}_c(j\omega_i)=c_ie^{jp_i}=c_i(\cos p_i+j\sin p_i),\ i=1,2 \quad (5)$$

Eq. (5) serves as the step stone to our analytical solution for designing lag/lead or general second order compensators.

Note that in most of control textbooks [10-16], among margins GM, PM and crossover frequencies ω_1, ω_2, only two or three of them are specified in compensator design. This is mainly due to two considerations. First, some of those parameters are not provided in actual design; second, and most importantly, the conventional trial-and-error graphical method does not allow all four parameters to be specified. In this paper, however, we assume all four are given and other cases can be considered easily based on our analytical solution, as indicated for a special class of three-parameter compensators in our previous work [8, 9].

3. SINGLE LAG OR LEAD COMPENSATORS

A single phase lag or lead compensator is expressed as,

$$\overline{G}_c=\frac{1+\alpha\tau s}{1+\tau s} \quad (6)$$

where $\alpha<1$ or $\alpha>1$ indicates a lag or lead compensator, respectively. Based on specification (2) or (4), we have,

$$\left|\overline{G}_c(j\omega)\right|=c\ ,\ \angle\overline{G}_c(j\omega)=p \quad (7)$$

where (c, p, ω_c) is either $(c1, p1, \omega_1)$ or $(c2, p2, \omega_2)$.

As indicated by Eq. (5), comparing the real and imaginary parts of Eq. (7), we can find easily that,

$$\alpha=\frac{c(c-\cos p)}{c\cos p-1},\ \tau=\frac{c\cos p-1}{c\omega\sin p} \quad (8)$$

Since $-\pi/2<p<\pi/2$, we have $\cos p>0$, thus,

$$\cos p=1/\sqrt{1+\delta^2}$$

where $\delta=\tan(p)$. Therefore, in terms of δ,

$$\alpha=\frac{c(c\sqrt{1+\delta^2}-1)}{c-\sqrt{1+\delta^2}}\ ,\ \tau=\frac{c-\sqrt{1+\delta^2}}{c\delta\omega} \quad (9)$$

which are identical with the result given in [8], however, the derivation process in [8] is much more complicated and involves nonlinear and transcendental equations.

From [8], the lead/lag compensation theorem for single phase lead or lag compensators can be stated as,

a) A single phase lead compensator exists if and only if,

$$\sqrt{1+\delta^2}<c,\ \delta>0,$$

b) A single phase lag compensator exists if and only if,

$$\sqrt{1+\delta^2}<1/c\ ,\ \delta\ <\ 0\ .$$

4. THREE-PARAMETER LAG-LEAD COMPENSATORS

A special class of three-parameter lag-lead compensators has been proposed and studied in [6, 9], they can be expressed as,

$$\overline{G}_c(s)=\frac{1+\alpha\tau s}{1+\tau s}\cdot\frac{1+\beta\sigma s}{1+\sigma s} \quad (10)$$

where $\tau>0,\ \sigma>0,\ \alpha>0,\ \beta>0.$ Parameters α and β are related by $\alpha\beta=1$.

Substitute s with $j\omega$ in Eq. (10), we have,

$$\overline{G}_c(j\omega)=\frac{1+j\Delta(\omega)\Gamma}{1+j\Delta(\omega)}$$

where,

$$\Gamma=\frac{\alpha\tau+\beta\sigma}{\tau+\sigma},\ \Delta(\omega)=\frac{(\tau+\sigma)\omega}{1-\tau\sigma\omega^2}$$

Based on Eq. (5), we find that,

$$\Gamma(c,\delta)=\frac{c(c-\cos p)}{c\cos p-1},\ \Delta(c,\delta)=\frac{c\cos p-1}{c\omega\sin p} \quad (11)$$

or in terms of δ,

$$\Gamma(c,\delta)=\frac{c\left(c\sqrt{1+\delta^2}-1\right)}{c-\sqrt{1+\delta^2}},\ \Delta(c,\delta)=\frac{c-\sqrt{1+\delta^2}}{c\delta}$$

which are identical with the result given in [9], again, the derivation process is much simpler here.

From Γ and Δ, we can find $(\tau,\sigma,\alpha,\beta)$ easily, as one can see in [9] or from the next section.

5. GENERAL LAG/LEAD COMPENSATORS

In general, for a lag-lag, lag-lead, or lead-lead serial combination, we have,

$$\begin{aligned}\overline{G}_c(s)&=\frac{1+\alpha\tau s}{1+\tau s}\cdot\frac{1+\beta\sigma s}{1+\sigma s}\\&=\frac{1+(\alpha\tau+\beta\sigma)s+\alpha\tau\beta\sigma s^2}{1+(\tau+\sigma)s+\tau\sigma s^2}\end{aligned} \quad (12)$$

where $\tau>0,\ \sigma>0,\ \alpha>0,\ \beta>0.$

Substitute s with $j\omega$ in Eq. (12), we have,

$$\overline{G}_c(j\omega)=\frac{\Omega_1(\omega)+j\Lambda_2}{\Omega_3(\omega)+j\Lambda_4}$$

where,

$$\Lambda_1=\alpha\tau\beta\sigma,\ \Lambda_2=\alpha\tau+\beta\sigma,\ \Lambda_3=\tau\sigma,\ \Lambda_4=\tau+\sigma,$$

$$\Omega_1(\omega)=\frac{1-\Lambda_1\omega^2}{\omega},\ \Omega_3(\omega)=\frac{1-\Lambda_3\omega^2}{\omega} \quad (13)$$

Define,

$$A_i=c_i\cos p_i\ ,\ B_i=c_i\sin p_i,\ i=1,2$$

then, Eq. (5) leads to the following linear equations,

$$\begin{aligned}&M(\omega_1,\omega_2,A_1,B_1,A_2,B_2)\Lambda\\&=N(\omega_1,\omega_2,A_1,B_1,A_2,B_2)\end{aligned} \quad (14)$$

where

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{bmatrix}, \quad M = \begin{bmatrix} \omega_1 & 0 & -A_1\omega_1 & -B_1 \\ 0 & 1 & B_1\omega_1 & -A_1 \\ \omega_2 & 0 & -A_2\omega_2 & -B_2 \\ 0 & 1 & B_2\omega_2 & -A_2 \end{bmatrix}, \quad N = \begin{bmatrix} (1-A_1)/\omega_1 \\ B_1/\omega_1 \\ (1-A_2)/\omega_2 \\ B_2/\omega_2 \end{bmatrix}$$

Once Λ is found from Eq. (14), $(\tau, \sigma, \alpha, \beta)$ can calculated by solving two quadratic equations as,

$$(\tau, \sigma) = (\Lambda_4 \pm \sqrt{\Lambda_4^2 - 4\Lambda_3})/2\,; \qquad (15)$$

$$(\alpha\tau, \beta\sigma) = (\Lambda_2 \pm \sqrt{\Lambda_2^2 - 4\Lambda_1})/2 \qquad (16)$$

where plus '+' is for τ and α and minus '-' is for σ and β. Note that actually this process will lead to four solutions, corresponding to

$(\tau, \sigma, \alpha, \beta)$,

$(\sigma, \tau, \beta, \alpha)$,

$(\tau, \sigma, \beta\sigma/\tau, \alpha\tau/\sigma)$, and

$(\sigma, \tau, \alpha\tau/\sigma, \beta\sigma/\tau)$

respectively. Clearly, all four lead to the same final combination for lag/lead compensation.

6. General Second-Order Compensators

Consider a general second compensator in the form of,

$$\overline{G}_c(s) = \frac{a_2 s^2 + a_1 s + 1}{b_2 s^2 + b_1 s + 1}\,, \qquad (17)$$

where $a_i b_i > 0$, $i = 1,2$.

Substitute s with $j\omega$ in Eq. (17), from Eq. (5), we have,

$$M(\omega_1, \omega_2, A_1, B_1, A_2, B_2)\chi = N(\omega_1, \omega_2, A_1, B_1, A_2, B_2) \qquad (18)$$

where M and N is same as in the previous section, and

$$\chi = [a_1 \quad a_2 \quad b_1 \quad b_2]^T\,.$$

Solving linear equations (18), we find the compensator immediately. Generally, compensator in (17) can not be factorized into a series of two lead or lag compensators.

To consider the case that the original point is a pole for compensation, such as the case of PID compensation, we can assume a new form for $\overline{G}_c(s)$,

$$\overline{G}_c(s) = \frac{a_2 s^2 + a_1 s + 1}{s(b_2 s^2 + b_1 s + 1)}\,.$$

In this case, the compensator can be determined from Eq. (18) by replacing A_i and B_i according to the following rule,

$$A_i \to -\omega_i B_i,\; B_i \to \omega_i A_i\,.$$

7. Conclusions

In this paper, we present a set of analytic solutions to various compensator design problems in the classical control analysis and design in frequency domain. Based those analytic solutions, we can eliminate completely the need for the conventional graphic-based trial-and-error method used for such problems and practiced by students in classrooms and engineers in industry since the very begin of control as an independent disciplinary, as one can see from all the contemporary textbooks in control. Related textbook and educational issues will be addressed elsewhere.

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